M624 HOMEWORK – SPRING 2026

Prof. Andrea R. Nahmod

SET 1 - Due Tuesday 02/17/2026 by 5pm

From Chapter 4 :

Additional Problem 1: Prove the Auxiliary Lemma we stated in Lecture 2. Namely prove the following:

A normed vector space $(X, \|\cdot\|)$ is *complete* if and only if every absolutely convergent series in X converges in X.

Recall; i) A series $\{x_n\}_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.
ii) A series $\sum_{n=1}^{\infty} x_n$ converges in X if there exists and x in X such that $\lim_{N\to\infty} \sum_{n=1}^{N} x_n = x$ in X; that is if $\| \sum_{n=1}^{N} x_n - x\| \to 0$ as $N \to \infty$

Additional Problem 2: For any $1 \leq p < \infty$ consider the space

$$L^p(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C},\ \text{measurable},: \|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p\, dm\right)^{\frac{1}{p}} < \infty\}.$$

Assume that $\|f\|_{L^p(\mathbb{R}^d)}$ is a norm† whence $d_p(f,g) := \|f-g\|_{L^p(\mathbb{R}^d)}$ defines a metric and L^p is a metric space. Use the proof of completeness that I gave in class from Folland (Lecture 3) when $p = 2$ to **prove** that $L^p(\mathbb{R}^d)$ is *complete*.

(†) Question: can you guess what would you need to prove the triangle inequality when $p \neq 2$?. Justify your answer.

From Chapter 4 (pp 193-194): 1, 2*, 3, 4 (show completeness only), 5, 6, 7.

$*$ to show that $f-g$ is *orthogonal* to g you need to show that $\langle f-g, g\rangle = 0$.

SET 2 - Due Thursday 02/26/2026 by 5pm

From Chapter 4 (pp 195-197): 8a)

Pb. I. (This is an undergrad. problem but a very useful property to remember)
Let $\phi : \mathbb{R} \to \mathbb{R}$ be a periodic function with period p; that is $\phi(x+p) = \phi(x)$, $\forall x \in \mathbb{R}$. Assume that ϕ is integrable on any finite interval.

(a) Prove that for any $a, b \in \mathbb{R}$

$$\int_a^b \phi(x)dx = \int_{a+p}^{b+p} \phi(x)dx = \int_{a-p}^{b-p} \phi(x)dx$$

(b) Prove that for any $a \in \mathbb{R}$

$$\int_{-p/2}^{p/2} \phi(x+a)dx = \int_{-p/2}^{p/2} \phi(x)dx = \int_{-p/2+a}^{p/2+a} \phi(x)dx$$

In particular we have that $\int_a^{a+p} \phi(x)\, dx$ does not depend on a, as we discussed in class.

Pb.II. Consider $f \in L^2([-\pi, \pi])$ and <u>assume</u> that $\sum_{n\in\mathbb{Z}} a_n e^{inx} = f(x)$ a.e. x. Show that on any subinterval $[a, b] \subset [-\pi, \pi]$,

$$\int_a^b f(x)\, dx = \sum_n \int_a^b a_n\, e^{inx}\, dx.$$

In particular if $g(x) = \int_a^x f(y)dy$, the Fourier coefficients and series of $g(x)$ can be obtained from a_n, the Fourier coefficients of f.

Pb. III. For $0 < \alpha < 1$, we say that a function f is C^α-Hölder continuous with exponent α if there exists a constant $c = c_\alpha > 0$ such that $|f(x) - f(y)| \leq c\,|x - y|^\alpha$ for all x, y. For $k \in \mathbb{N}$, we can also define the space $C^{k,\alpha}$ to be that of functions which are k-th times differentiable and whose k-th derivative is C^α-Hölder continuous (we could relabel C^α as $C^{0,\alpha}$). Consider now f a 2π-periodic $C^{k,\alpha}$ function. If a_n are the Fourier coefficients of f, show that for some $C > 0$ independent of n,

$$|a_n| \leq \frac{C}{|n|^{k+\alpha}}$$

<u>Hint</u> Start by integrating by parts (and using periodicity).

<u>*Bonus Problem (do but do not turn in)*</u> : 2*a)b) from Chapter 4, pp 202.

<u>SET 3 - Due Thursday 03/12/2026 by 5pm</u>

From Chapter 4 (pp 195-197): 10, 11, 12, 13

Pb. I. Consider the subspace $\mathcal{S}$ of $L^2([0,1])$ spanned by the functions: 1, x, and x^3.
a) Find an orthonormal basis of $\mathcal{S}$.
b) Let $P_{\mathcal{S}}$ denote the orthogonal projection on the subspace $\mathcal{S}$, compute $P_{\mathcal{S}} x^2$.

Assigned Reading from Chapter 4 of [Stein-Shakarchi Vol 3]:

a) Examples 1) and 2) on pages 178–180.

b) Please read ahead sections 4.5.1, 4.5.2 and 4.5.3 of section 5 chapter 4.

From Chapter 4 (pp 195-197): 18, 19, 20

SET 4 - Due Thursday 03/26/2026 by 5pm

Assigned Reading from Chapter 4 of [Stein-Shakarchi Vol 3]:
Remarks (a), (b), (c) on pages 184-185.

From Chapter 4 (pp 187): Read and rewrite filling in all details the Proof of Proposition 5.5 (do but do not turn in)

From Chapter 4 (pp 189): Read and rewrite filling in all details the Proof of Proposition 6.1 (do but do not turn in).

From Chapter 4 (pp 197-202): 21, 22, 23, 25, 26.

SET 5 - Due Thursday 04/02/2026 by 5pm

From Chapter 4 (pp 197-202): 28 (for c) consider only the *if* part), 30, 32, 33.

Do but do not turn in. From Chapter 4: 29 (p 199-200) and 6 in Section 8 (p. 203-204). The Fredholm's Alternative for compact operators holds generally on Banach spaces. You may want to research this further; see for example Peter Lax's Functional Analysis, Reed and Simon's Methods of Modern Mathematical Physics, Vol. 1 or Extended version, Conway's A Course in Functional Analysis, etc.).