M523Honors: Introduction to Modern Analysis

Homework

Spring 2025

Note: **Do** but don't turn in (at least yet) the problems followed by an asterisk $*$.

Assignment 1. Due Thursday 02/13/2025.

From Section 1.1: 1, 3, 4, 5, 6a), 7, 10, 11.

From Section 1.2: 1, 2, 3, 5a), 5d), 6a), 6b), 6c), 7a), 7b), 7d), 9b), 10.

Additional Problem 1*: Show that $(\mathbb{C}, +, \cdot)$ is a field.

Assignment 2. Due Thursday 02/27/2025.

From Section 1.3: 1, 3, 5, 7, 8, 9.

Assignment 3. Due Thursday 03/13/2025.

From Section 1.4: 1, 4, 7, 8, 9, 10a)c) 11a)

(In 10c) f^2 means $[f(x)]^2$ and **not** $f \circ f$)

From Section 2.1: 1a)c), 2a)c)d), 3b)c)d), 4, 5, 6, 7

Assignment 4. Due Thursday 3/27/2025.

From Section 2.2: 1, 2, 3, 4, 5(modified), 7, 8, 9.

Problem 5(modified). Prove theorem 2.26 in the special case when $a_n = 1$. That is prove $\frac{1}{b_n} \to \frac{1}{b}$ under the same hypothesis.

Note: the general case of problem 5 is posted as a Handout in the class webpage. Check Handouts also for other Hints for sections 2.2 and 2.4.

From Section 2.4: 1, 3, 5, 6, 7

Assignment 5. Due Thursday 4/10/2025.

From Section 2.4: 2, 9, 10, 13.

From Section 2.5: 1, 3(modified), 4, 5, 7, 8.

Problem 3(modified). *Suppose a set S of real numbers is bounded and let η be a lower bound for S. Show that η is the greatest lower bound of S if and only if for every $\varepsilon > 0$ there is an element of S in the interval $[\eta, \eta + \varepsilon]$*

Extra Problem 2* for Section 2.5 Prove the existence of *greatest lower bounds* just as we proved the existence of *least upper bound* in Theorem 2.5.1

Special Project I:* Do Project #1 at the end of Chapter 2.

Assignment 6. Due Thursday 04/17/2025.

From Section 2.6: 1, 2, 3, 4, 6, 8, 9, 10, 11, 13.

Special Project II:* Do Problem 14 of Section 2.4.

Special Project III:* Do Project #5 at the end of Chapter 2 (page 70)

Assignment 7. Due Thursday 4/24/2025.

From Section 3.1 : 5, 7, 8b), 9, 10.

From Section 3.2 : 1, 3, 4, 7, 10(modify the statement to the interval $[0, 1]$ for simplicity).

Hint for 4): Consider $h(x) = f(x) - g(x)$ and use problem 3)

Special Project IV:* Do Problem 13 of Section 3.2

Assignment 8. Due Thursday 5/8/2025.

From Section 3.3: 2, 12a), 14

Hint for 14) argue by contradiction, suppose $f(x_0) > 0$ and use problem 10 section 3.1 to find a small interval I around x_0 such that for $x \in I$, you have $f(x) \geq \frac{f(x_0)}{2}$. Then select a_1, b_1 in this interval I to draw a contradiction).

From Section 5.1: 2, 7, 12.

From Section 5.2: 1, 2a)b), 6.

Hints For 5.1 #7: for each $n \geq 1$ choose an x in $[0,1]$ such that $n\,x = 1$. Call that x, x_n and compute $f_n(x_n)$.

For 5.1 #12: Given $\varepsilon > 0$, write $|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|$ and find $N = N(\varepsilon)$ so that (a) the first term on the r.h.s of the inequality is less than $\varepsilon/2$ thanks to the *uniform convergence* of f_n to f; (b) the second term on the r.h.s of the inequality is less than $\varepsilon/2$ thanks to the *continuity* of f

For 5.2 #2*a*): first prove that the sequence of functions $f_n(x) = (x + \frac{1}{n})^2$ converges uniformly to the function $f(x) = x$ on $[0,1]$ as n goes to infinity. Then use Theorem 5.2.2 to compute. Proceed similarly for b) for a suitable f.

For 5.2 #6: Denote by f the limiting function and write $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)|$.
First note that since the convergence is uniform on $[0,1]$, f must be continuous (why?) and hence bounded (why?). Second, prove that there exists N (think of $\varepsilon = 1$) such that for all $n \geq N$ the first term on the right hand side is less than 1. Third, note that each of the remaining functions f_n (with $1 \leq n \leq N-1$) is continuous and bounded on $[0,1]$ and there are only a finite number (namely $N-1$) of them. Finally, put all the ingredients together to conclude!

Special Project V:* Do Problem 15 of Section 3.3

Extra Assignment 9: Do to learn but do not to turn in.

First read/review on your own the results in 6.1 & 6.2. Then do.

From Section 6.3 1, 2, 6, 7, 9.