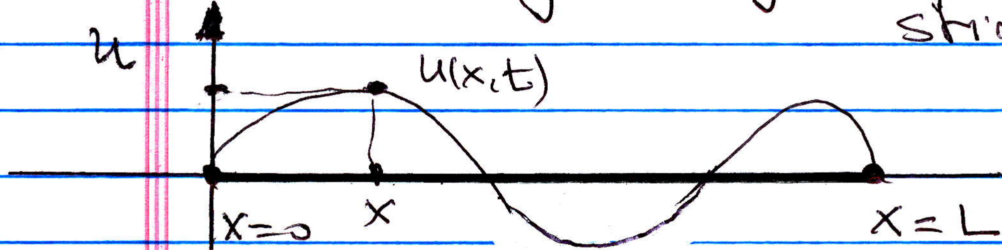
• DERIVATION OF WAVE EQUATION in 1 Dimension

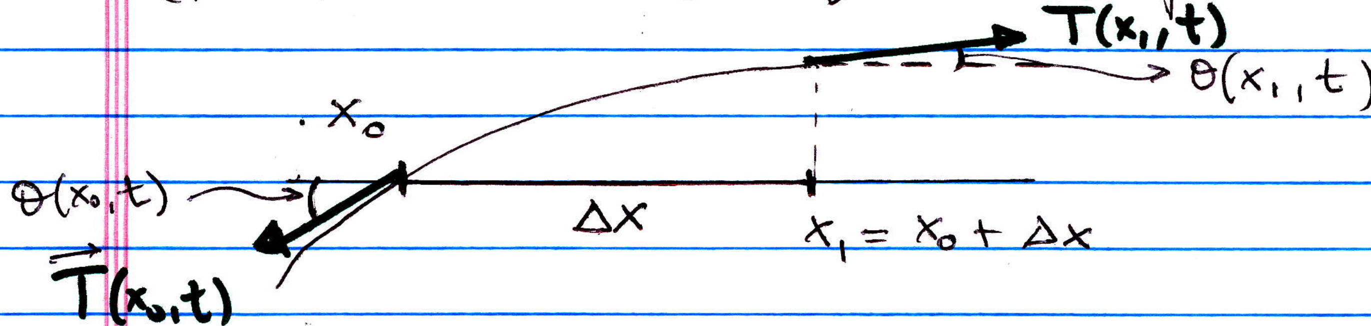
• **Vibrating String**: flexible homogeneous string of length L



{ SNAPSHOT AT time t fixed

• Assume the string is vibrating on the (x,u)-plane. Then $\underline{u(x,t)}$ = DISPLACEMENT FROM EQUILIBRIUM (REST POSITION) AT $\underline{\text{time } t}$ and $\underline{\text{position } x}$

Look at an infinitesimally piece of string (far from the endpoints). FOR example:



• String is perfectly $\underline{\text{FLEXIBLE}}$ $\Rightarrow$ $\underline{\text{tension}}$ (force) is $\underline{\text{DIRECTED}}$ $\underline{\text{TANGENTIALLY}}$ $\underline{\text{along the string}}$

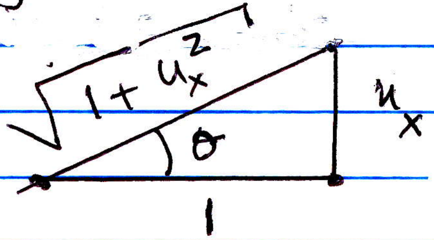
Let $T(x,t)$ = MAGNITUD OF THIS TENSION VECTOR

Let ρ = density (mass per unit length) of string (string homogeneous $\Rightarrow$ ρ = constant)

RECALL $\underline{\text{NEWTON's } 2^{nd} \text{ LAW}}$: "$F = m \cdot a$"

On the other hand that the projections of T onto its vertical and horizontal components are respectively T, sin θ and T. cos θ and that

(θ is angle as in drawing in page 1)



u_x = SLOPE OF STRING at x.

Then by Newton's second law we have for the horizontal and vertical forces that

$$(1)\quad \left.\frac{T \cdot 1}{\sqrt{1+u_x^2}}\right|_{x=x_0}^{x=x_1} = 0$$

since there is NO MOTION in the longitudinal direction.

$$(2)\quad \left.\frac{T u_x}{\sqrt{1+u_x^2}}\right|_{x=x_0}^{x=x_1} = \int_{x_0}^{x_1} \rho\, u_{tt}$$

VERTICAL ACCELERATION

SUM OF Vertical FORCES AT $x=x_0$ AND $x=x_1$

[NOTE VECTORS ARE IN OPPOSITE DIRECTIONS AT EACH END $x=x_0$ AND $x=x_1$.

Now we assume the motion is small i.e. that $|u_x|$ is small $\Rightarrow \sqrt{1+u_x^2} \approx 1$ (by Taylor's Exp.)

Then Eq. (1) says that T is CONSTANT along the string $\Rightarrow$ T is independent of x.

- Assume T is also independent of t (reasonable assumption)

Then Eq. (2) says:

$$(T u_x)_x = \rho u_{tt}$$

$$\Rightarrow \quad \boxed{u_{tt} = c^2 u_{xx}} \qquad c = \sqrt{\frac{T}{\rho}}$$

(= WAVE SPEED)